Optical soliton solutions of the quintic complex Swift-Hohenberg equation

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1. Introduction. Complicated optical systems with gain and loss can be described by the complex Swift-Hohenberg (S-H) equation. These include synchronously pumped optical parametric oscillators, pattern formation in convection cells and passively mode-locked lasers with special spectral features [1]-[3]. The main difference between the complex S-H equation and previously-studied master equations lies in its more involved spectral filtering term. At the same time, these complications do not allow us to analyse the solutions easily. In fact, it was not clear that such solutions may exist at all [1]-[3]. In this work, we study the quintic complex S-H equation in 1D and report various new exact solutions.

The normalized quintic complex S-H equation is[4]:

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\,\psi + \,(h + is)\psi_{tttt} + (\nu\, - i\mu)|\psi|^4\psi = i\delta\psi + i\beta\psi_{tt} + i\epsilon|\psi|^2\psi.$$

In mode-locked laser applications, z is the propagation distance or the cavity round-trip number (treated as a continuous variable), t is the retarded time, D is the 2nd order dispersion, h is the 4th order dispersion, ϵ is a nonlinear gain (or 2-photon absorption if negative) and δ (usu. negative) represents a constant gain or loss. The band-limited gain (e.g. due to an EDFA, where the gain band may be about 30 nm around 1.5 microns) is represented by β (parabolic spectrum shape) and s (4th order correction). We find exact forms for a range of solitons, including bright and dark cases, and both chirped and unchirped forms. They have various features which differentiate them from solitons of the complex Ginzburg-Landau equation [5]. These first solutions can give some clues for analysing more involved solutions.

2. Analysis. Let $\psi = f(t)\,exp[-i\,\Omega\,z\,]$. This reduces the quintic complex S-H

equation to an ordinary differential eqn. in f:

$$\Omega f + (\frac{D}{2} - i\beta)\, f''(t) + (1 - i\,\epsilon)|f|^2 f - i\delta f + (h + is)\, f''''(t) + (\nu - i\mu)|f|^4 f = 0. \quad (1)$$

We will show that basic [unchirped] bright and dark solitons exist, and that there are also chirped bright and dark solitons.

3.1 Bright soliton If we take $f = c\,g\,sech(g\,t)$, then we note that $f''(t)/[g^2 f(t)] = 2\,tanh^2(g\,t) - 1$ and that $f''''(t)/[g^4 f(t)] = 5 - 28\,tanh^2(g\,t) + 24\,tanh^4(g\,t)$. On dividing through by f, it is clear that the coefficients of $tanh^n(gz)$ for $n = 0, 2, 4$ can be set to zero to reduce the problem to an algebraic one and obtain the solution. From the $n = 4$ term , we find a consistency condition on the equation parameters: $\nu = -h\,\mu\,/s$. Thus $c^4 = 24s/\mu$ $(= -24h/\nu)$. From the other terms we find g, c, etc and conditions we need to impose as constraints on the equation parameters. Thence we can easily find β, δ, g and Ω Note that , if $\epsilon = 0$ the solution simplifies, as then $g = \frac{\beta}{10s}$, so

$$f = \sqrt{\frac{\beta}{5}}\,(\frac{6}{s\,\mu})^{\frac{1}{4}}\; sech[\sqrt{\frac{\beta}{10s}}\,t].$$

3.2 Black soliton. Here we take If we take $f = c\,g\,tanh(g\,t)$, then we see that $f''(t)/[g^2 f(t)] = 2\;[tanh^2(gt) - 1]$, and that $f''''(t)/[g^4 f(t)] = 8\;[tanh^2(gt) - 1]\,[3\,tanh^2(gt) - 2]$. Again, on dividing by f, it is clear that the coefficients of $tanh^n(gt)$ for $n = 0, 2, 4$ can be set to zero to reduce the problem, and we then determine all the solution parameters .

4.1. Chirped bright soliton. This involves the function $f = a(t)\,exp[id\,log\,a(t)]$, where $a(t) = gc\,sech(g\,t)$. With this form, the derivatives can still be written in a convenient way. For example, $f''(t)/[g^2 f(t)] = -\,(d - i)\,[i + (d - 2i)\,tanh^2(gt)]$.

We need to find the roots of a 4th order polynomial in d:

$$(\mu\,h + s\,\nu)(d^4 - 35d^2 + 24) + 10d(\nu\,h - s\,\mu)(5 - d^2) = 0. \quad (2)$$

Thus d depends on a balance of the highest order derivative (4th) and strongest nonlinearity and is quite different from that of the CGLE [5] or CGLE with an integral term [6] where d is determined by β and ϵ. Eqn(2) provides insight into the chirpless $(d = 0)$ case, as we see that $d = 0$ is a root of eqn.(2) when $\mu\,h + s\,\nu = 0$.

4.2 Chirped black soliton Here $f(t) = a(t)\,exp[id\,log\,b(t)]$, where $a(t) = r\,g\,tanh(g\,t)$ and $b(t) = g\,sech(g\,t)$, and we proceed as above to find the solution.

5. Energy and momentum balance. As any other equation describing dissipative systems, the quintic complex S-H equation does not have any conserved quantities.

Instead, we can write balance equations for the energy and momentum. A study of the quintic complex S-H eqn., using the energy balance approach ([5],[6]) leads to the following evolution equation for the energy:

$$\frac{d}{dz}\int_{-\infty}^{\infty}|\psi|^2 dt = 2\int_{-\infty}^{\infty}\Big[\delta|\psi|^2 - \beta\Big|\frac{\partial\psi}{\partial t}\Big|^2 - \,s\,|\psi_{tt}|^2 + \epsilon|\psi|^4 + \mu|\psi|^6\Big]dt. \tag{3}$$

By definition, the right hand side of this equation is the rate of change of the energy. The term with s here is new in comparison with similar equation for CGLE [5]. The form of this term reflects its role as a higher-order band-limited gain. For any stationary exact solution, we need the r.h.s. of eqn.(3), and also the rate of change of the momentum, to be zero. This provides a way of finding or checking solutions. The balance equation is also an important tool in studying the interaction between the pulses [5].

Conclusion. Solutions presented here are novel examples of exact solutions which exist for the quintic complex S-H equation. As for the Ginzburg -Landau equation [5] they certainly do not cover the whole set of possible solutions. In fact, they only represent a small subset of a variety of soliton-like solutions. Other solutions have to be studied numerically and this may take considerable amount of simulations. However, finding the exact solutions is an important step in analysing the laser system with more involved spectral properties which is described by the quintic complex S-H equation.

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